

UNIFORM ANALYTIC PROPERTIES OF REPRESENTATION ZETA FUNCTIONS OF FINITELY GENERATED NILPOTENT GROUPS

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ABSTRACT. Let G be a finitely generated nilpotent group. The representation zeta function $\zeta_G(s)$ of G enumerates twist isoclasses of finite-dimensional irreducible complex representations of G . We prove that $\zeta_G(s)$ has rational abscissa of convergence $\alpha(G)$ and may be meromorphically continued to the left of $\alpha(G)$ and that, on the line $\{s \in \mathbb{C} \mid \operatorname{Re}(s) = \alpha(G)\}$, the continued function is holomorphic except for a pole at $s = \alpha(G)$. A Tauberian theorem yields a precise asymptotic result on the representation growth of G in terms of the position and order of this pole.

We obtain these results as a consequence of a result establishing uniform analytic properties of representation zeta functions of torsion-free finitely generated nilpotent groups of the form $\mathbf{G}(\mathcal{O})$, where \mathbf{G} is a unipotent group scheme defined in terms of a nilpotent Lie lattice over the ring \mathcal{O} of integers of a number field. This allows us to show, in particular, that the abscissae of convergence of the representation zeta functions of such groups and their pole orders are invariants of \mathbf{G} , independent of \mathcal{O} .

1. INTRODUCTION

1.1. Main results. In this paper we study zeta functions associated to finitely generated nilpotent groups. Let G be such a group and let ρ, σ be complex representations of G . We say that ρ and σ are *twist-equivalent* if there exists a 1-dimensional representation χ of G such that $\rho \otimes \chi \cong \sigma$. Twist-equivalence is an equivalence relation on the set of finite-dimensional irreducible complex representations of G . Its classes are called *twist isoclasses*. The number $\tilde{r}_n(G)$ of twist isoclasses of G of dimension n is finite for every $n \in \mathbb{N}$; cf. [16, Theorem 6.6].

The (*representation*) *zeta function* of G is defined to be the Dirichlet generating function

$$\zeta_G(s) := \sum_{n=1}^{\infty} \tilde{r}_n(G) n^{-s},$$

where s is a complex variable. The sequence $(\tilde{r}_n(G))_{n \in \mathbb{N}}$ grows polynomially and hence $\zeta_G(s)$ converges on a complex half-plane $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > \alpha\}$ for some $\alpha \in \mathbb{R}$;

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cf. [25, Lemma 2.1]. The infimum of such α is the *abscissa of convergence* $\alpha(G)$ of $\zeta_G(s)$. Note that if $\tilde{r}_n(G) \neq 0$ for infinitely many $n \in \mathbb{N}$, then

$$\alpha(G) = \limsup_{N \rightarrow \infty} \frac{\log \sum_{n=1}^N \tilde{r}_n(G)}{\log N}.$$

Hence $\alpha(G)$ determines the rate of polynomial growth of $\left(\sum_{n=1}^N \tilde{r}_n(G) \right)_{N \in \mathbb{N}}$.

We consider a class of torsion-free finitely generated nilpotent groups (or \mathcal{T} -groups, for short) obtained from unipotent group schemes over number rings. Let \mathcal{O}_K be the ring of integers of a number field K and Λ be an \mathcal{O}_K -Lie lattice of nilpotency class c and \mathcal{O}_K -rank h . By an \mathcal{O}_K -Lie lattice we mean a free and finitely generated \mathcal{O}_K -module, together with an antisymmetric, bi-additive form $[\cdot, \cdot]$ satisfying the Jacobi identity. If $c > 2$ we assume that $\Lambda' := [\Lambda, \Lambda] \subseteq c! \Lambda$. This ensures that the Hausdorff series may be used to associate to Λ a unipotent group scheme \mathbf{G}_Λ ; see [25, Section 2.1.2]. If $c = 2$, then the group scheme \mathbf{G}_Λ may be defined more directly and without the above assumption on Λ ; cf. [25, Section 2.4]. For every number field L containing K , with ring of integers \mathcal{O}_L , the group $\mathbf{G}_\Lambda(\mathcal{O}_L)$ is a \mathcal{T} -group of nilpotency class c and Hirsch length $h \cdot [L : \mathbb{Q}]$. The main result of this paper is the following. Given arithmetic functions $f(N)$ and $g(N)$, we write ' $f(N) \sim g(N)$ as $N \rightarrow \infty$ ' to indicate that $\lim_{N \rightarrow \infty} f(N)/g(N) = 1$.

Theorem A. *Let Λ and $\mathbf{G} = \mathbf{G}_\Lambda$ be as above. Then there exist $a(\mathbf{G}), \delta(\mathbf{G}) \in \mathbb{Q}_{>0}$, and $\beta(\mathbf{G}) \in \mathbb{N}$ such that, for all finite extensions L/K , the following hold.*

- (1) *The abscissa of convergence of $\zeta_{\mathbf{G}(\mathcal{O}_L)}(s)$ is $a(\mathbf{G})$, i.e. $a(\mathbf{G}) = \alpha(\mathbf{G}(\mathcal{O}_L))$.*
- (2) *The zeta function $\zeta_{\mathbf{G}(\mathcal{O}_L)}(s)$ has meromorphic continuation to the half-plane $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > a(\mathbf{G}) - \delta(\mathbf{G})\}$. On the line $\{s \in \mathbb{C} \mid \operatorname{Re}(s) = a(\mathbf{G})\}$ the continued zeta function is holomorphic except for a pole at $s = a(\mathbf{G})$, of order $\beta(\mathbf{G})$.*

Moreover, there exists $c(\mathbf{G}(\mathcal{O}_L)) \in \mathbb{R}_{>0}$ such that

$$\sum_{n=1}^N \tilde{r}_n(\mathbf{G}(\mathcal{O}_L)) \sim c(\mathbf{G}(\mathcal{O}_L)) N^{a(\mathbf{G})} (\log N)^{\beta(\mathbf{G})-1} \text{ as } N \rightarrow \infty.$$

Theorem A asserts that the invariants $a(\mathbf{G})$ and $\beta(\mathbf{G})$ are independent of L , whereas the number $c(\mathbf{G}(\mathcal{O}_L))$ may vary with L . Theorem A answers [25, Question 1.4] affirmatively.

Example 1.1. We illustrate Theorem A in the case of the Heisenberg group scheme \mathbf{H} associated to the Heisenberg \mathbb{Z} -Lie lattice of strict upper-triangular 3×3 -matrices. Let \mathcal{O} be the ring of integers of a number field L . The \mathcal{T} -group $\mathbf{H}(\mathcal{O})$ is isomorphic to the group of upper-unitriangular 3×3 -matrices over \mathcal{O} , of nilpotency class 2 and Hirsch length $3[L : \mathbb{Q}]$. The zeta function of $\mathbf{H}(\mathcal{O})$ is

$$(1.1) \quad \zeta_{\mathbf{H}(\mathcal{O})}(s) = \frac{\zeta_L(s-1)}{\zeta_L(s)} = \prod_{\mathfrak{p} \in \operatorname{Spec}(\mathcal{O})} \frac{1 - |\mathcal{O}/\mathfrak{p}|^{-s}}{1 - |\mathcal{O}/\mathfrak{p}|^{1-s}},$$

where $\zeta_L(s)$ is the Dedekind zeta function of L and \mathfrak{p} ranges over the nonzero prime ideals of \mathcal{O} . This is proved in [19] for $L = \mathbb{Q}$, in [10] for quadratic number fields, and in [25, Theorem B] for arbitrary number fields. The zeta function $\zeta_{\mathbf{H}(\mathcal{O})}(s)$ has abscissa of convergence $a(\mathbf{H}) = 2$ and may be meromorphically continued to the whole complex plane. The continued function has no singularities on the line $\{s \in \mathbb{C} \mid \operatorname{Re}(s) = 2\}$, apart from a simple pole at $s = 2$. Formula (1.1) implies that

$$\sum_{n=1}^N \tilde{r}_n(\mathbf{H}(\mathcal{O})) \sim \frac{1}{2\zeta_L(2)} N^2 \text{ as } N \rightarrow \infty.$$

We remark that $\zeta_L(2)$ depends subtly on L ; cf. [29, Theorem 1].

Remark 1.2. The part of statement (2) establishing the uniqueness of the pole at $s = a(\mathbf{G})$ on the line $\{s \in \mathbb{C} \mid \operatorname{Re}(s) = a(\mathbf{G})\}$ is owed to the fact that Artin L -functions are used to establish the meromorphic continuation. Whether there are other poles in the domain $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > a(\mathbf{G}) - \delta(\mathbf{G})\}$ is an interesting open question.

Theorem A yields uniform statements for families of \mathcal{T} -groups of a specific form, viz. those coming from nilpotent Lie lattices. In the following corollary we apply Theorem A and the methods underlying its proof to finitely generated nilpotent groups at large.

Corollary B. *Let G be a finitely generated nilpotent group.*

- (1) *The abscissa of convergence $\alpha(G)$ of $\zeta_G(s)$ is a rational number.*
- (2) *The zeta function $\zeta_G(s)$ has a meromorphic continuation to $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > \alpha(G) - \delta\}$ for some $\delta \in \mathbb{Q}_{>0}$. On the line $\{s \in \mathbb{C} \mid \operatorname{Re}(s) = \alpha(G)\}$, the continued zeta function has a unique pole at $s = \alpha(G)$, of order $\beta(G)$, say.*
- (3) *There exists $c(G) \in \mathbb{R}_{>0}$ such that*

$$\sum_{n=1}^N \tilde{r}_n(G) \sim c(G) N^{\alpha(G)} (\log N)^{\beta(G)-1} \text{ as } N \rightarrow \infty.$$

Moreover, the abscissa of convergence $\alpha(G)$ and the pole order $\beta(G)$ of the continued function are commensurability invariants: if $H \leq G$ is a subgroup of G of finite index in G , then $\alpha(G) = \alpha(H)$ and $\beta(G) = \beta(H)$.

Remark 1.3. The February 2015 version of [12] establishes part (1) (but not parts (2) and (3)) of Corollary B as an application of model-theoretic results on zeta functions associated to definable equivalence relations. We discuss this approach in some detail in Section 1.3.1.

1.2. Methodology. To prove Theorem A, we present $\zeta_{\mathbf{G}(\mathcal{O}_L)}(s)$ as a suitable Euler product and control some analytic properties of this product. The simplest such Euler product holds for general finitely generated nilpotent groups and is indexed by

the rational primes. Let G be a finitely generated nilpotent group. Then

$$(1.2) \quad \zeta_G(s) = \prod_{p \text{ prime}} \zeta_{G,p}(s),$$

where, for a prime p , the Euler factor $\zeta_{G,p}(s) := \sum_{i=0}^{\infty} \tilde{r}_{p^i}(G) p^{-is}$ enumerates the twist isoclasses of irreducible complex representations of G of p -power dimension. It may also be viewed as the zeta function of the pro- p completion \hat{G}^p of G , enumerating *continuous* irreducible complex representations of \hat{G}^p up to twists by continuous one-dimensional such representations. Each Euler factor is a rational function in p^{-s} ; cf. [12, Theorem 1.5].

In the situation of Theorem A, these features may be refined in the following sense. Let Λ and $\mathbf{G} = \mathbf{G}_{\Lambda}$ be as in the theorem. Fix a finite extension L of K and write $\mathcal{O} = \mathcal{O}_L$ for its ring of integers. For a nonzero prime ideal \mathfrak{p} of \mathcal{O} we denote by $\mathcal{O}_{\mathfrak{p}}$ the completion of \mathcal{O} at \mathfrak{p} , with residue field cardinality q and residue field characteristic p . The zeta function $\zeta_{\mathbf{G}(\mathcal{O})}(s)$ is the Euler product

$$(1.3) \quad \zeta_{\mathbf{G}(\mathcal{O})}(s) = \prod_{\mathfrak{p} \in \text{Spec}(\mathcal{O})} \zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s)$$

ranging over all nonzero prime ideals \mathfrak{p} of \mathcal{O} , where $\zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s)$ enumerates the *continuous* irreducible complex representations of the finitely generated nilpotent pro- p group $\mathbf{G}(\mathcal{O}_{\mathfrak{p}})$ up to twists by continuous one-dimensional representations; cf. [25, Proposition 2.2]. Clearly, the Euler product (1.3) refines (1.2). For almost all (i.e. all but finitely many) \mathfrak{p} , the local zeta function $\zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s)$ is a rational function in q^{-s} and satisfies a functional equation upon inversion of q ; see [25, Theorem A]. Key to this general result is a presentation of almost all of the Euler factors $\zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s)$ by *formulae of Denef type*, i.e. finite sums of the form

$$(1.4) \quad \sum_{i=1}^r |\overline{V}_i(\mathfrak{o}/\mathfrak{p})| W_i(q, q^{-s}).$$

Here, the terms $|\overline{V}_i(\mathfrak{o}/\mathfrak{p})|$ denote the numbers of $\mathfrak{o}/\mathfrak{p} \cong \mathbb{F}_q$ -rational points of the reductions modulo \mathfrak{p} of suitable algebraic varieties V_i defined over \mathcal{O}_K and $W_i(X, Y) \in \mathbb{Q}(X, Y)$ are bivariate rational functions enumerating integral points in polyhedral fans. Expressions such as (1.4) always allow for a common denominator which is a product of factors of the form $1 - q^{-as-b}$, for suitable integers a, b . Expressing Euler factors through formulae of the form (1.4) shows that they are *uniformly rational* in q^{-s} in the sense of [12, Section 6].

Our proof of Theorem A proceeds by analysing Euler products of formulae of Denef type arising in the specific context of representation zeta functions of finitely generated nilpotent groups. Here, such formulae are facilitated by the Kirillov orbit method, which allows for a parametrization of twist isoclasses of finite-dimensional representations by co-adjoint orbits. Roughly speaking, we show that the Euler products of formulae of Denef type representing almost all factors in the Euler product (1.3) can be approximated by products of translates of the Artin L -functions to

continue the relevant Euler product to the left of its abscissa of convergence. This analysis is motivated by and uses tools from [8], but extra care has to be taken to adapt this analysis to the specific kind of Denef formulae arising and to render it uniform over all extensions L of K . We adapt methods from [3] to show that the finitely many Euler factors not covered by our generic arguments do not affect the relevant analytic properties of the cofinite Euler product. A Tauberian theorem allows for the asymptotic final statement.

The methods we employ are inspired by but different from both those used to study the analytic properties of Euler products of \mathfrak{p} -adic *cone integrals* arising in the theory of subgroup growth of \mathcal{T} -groups and those used to approximate Euler products arising in the representation growth of arithmetic subgroups of semisimple groups. We discuss these classes of \mathfrak{p} -adic integrals in some detail in Sections 1.3.3 and 1.3.2.

Corollary B, which pertains to general finitely generated nilpotent groups, is obtained by applying Theorem A to a subgroup of finite index which fits into its remit. The existence of such subgroups follows from the Mal'cev correspondence.

1.3. Background and related research.

1.3.1. Representation zeta functions of nilpotent groups. The conclusions of Theorem A and Corollary B were previously observed in special cases. In [25, Theorem B], for instance, three infinite families of group schemes generalizing the Heisenberg group scheme (see Example 1.1) were studied. The explicit formulae given there for the zeta functions of the members of these families in terms of Dedekind zeta functions show that the respective abscissae of convergence are integers and independent of the number field. The respective constants in the asymptotic expansions, however, are of an arithmetic nature, being expressible in terms of special values of Dedekind zeta functions; see [25, Corollary 1.3]. Further examples of representation zeta functions of \mathcal{T} -groups, illustrating Theorem A and Corollary B, were computed in [9, 24]. It is not hard to show that every positive rational number is the abscissa of convergence of some \mathcal{T} -group of nilpotency class 2; see [24, Theorem 4.22]. It is an interesting open problem to characterize analytic data such as the invariants $a(\mathbf{G})$, $\delta(\mathbf{G})$, and $\beta(\mathbf{G})$ in Theorem A in terms of structural invariants of the group scheme. Note that Theorem A makes no claim as to the maximal value of $\delta(\mathbf{G})$. In the examples discussed, $\delta(\mathbf{G})$ can be chosen arbitrarily large. We expect it to be easy, however, to exhibit examples of \mathcal{T} -groups whose representation zeta functions do not have meromorphic continuation to the whole plane, but where this continuation has a natural boundary.

We already mentioned the fundamental contributions made in [12]. This paper – which developed substantially between its first posting in 2006 and its latest revision in 2015, acquiring two additional authors on the way – studies the more general setup of zeta functions enumerating classes of definable equivalence relations, a model-theoretic notion. This class of zeta functions contains representation zeta functions of finitely generated nilpotent groups. Already in its first version, by Hrushovski and Martin, [12] established the rationality in p^{-s} of *all* factors in the coarse Euler

product (1.2). The 2015-version established formulae for *almost all* of the fine Euler factors in (1.3) which are very similar to (1.4) (cf. [12, eq. (8.6)]) and proved the rationality of the abscissa of convergence of the representation zeta function of an arbitrary finitely generated nilpotent group as part of [12, Theorem 1.5]. (It appears that the methods of [12] might be sufficient to establish rationality of *all* Euler factors in (1.3), though this is not made explicit in [12].) The paper neither implies the other statements of Corollary B nor the uniformity statements of Theorem A. In fact, (uniform) statements about meromorphic continuation are not expected to hold in the general setting of zeta functions associated to definable equivalence relations. Rather, their validity seems to reflect specific features of representation growth of finitely generated nilpotent groups; cf. also [12, Remark 6.6].

In [20], Rossmann introduces and studies so-called topological representation zeta functions associated to unipotent algebraic group schemes such as the ones considered in the present paper. Informally, these rational functions are obtained as limits of local representation zeta functions as ' $p \rightarrow 1$ '; see [20] for further details and results.

1.3.2. Representation growth of 'semisimple' arithmetic groups. The theory of representation growth of groups is arguably most developed for arithmetic subgroups of semisimple algebraic groups over number fields. To be more precise, let \mathbf{G} be a semisimple group scheme defined over a number field K , together with a fixed embedding $\mathbf{G} \hookrightarrow \mathrm{GL}_d$ for some $d \in \mathbb{N}$. Consider, for simplicity's sake, the arithmetic group $\Gamma = \mathbf{G}(\mathcal{O}_S)$, where \mathcal{O}_S the ring of S -integers \mathcal{O}_S of K for some finite set of places S of K , including all the archimedean ones. If Γ satisfies the (strong) Congruence Subgroup Property (CSP), then the sequence $(r_n(\Gamma))_{n \in \mathbb{N}}$, enumerating the n -dimensional irreducible complex representations of Γ , grows at most polynomially ([17]) and the zeta function $\zeta_\Gamma(s) := \sum_{n=1}^{\infty} r_n(\Gamma) n^{-s}$ is an Euler product of the form

$$\zeta_\Gamma(s) = \zeta_{\mathbf{G}(\mathbb{C})}(s)^{|K:\mathbb{Q}|} \prod_{v \notin S} \zeta_{\mathbf{G}(\mathcal{O}_v)}(s);$$

cf. [15, Proposition 1.3]. Here, $\zeta_{\mathbf{G}(\mathbb{C})}(s)$ denotes the *Witten zeta function*, enumerating the finite-dimensional rational representations of the algebraic group $\mathbf{G}(\mathbb{C})$, whereas each non-archimedean factor $\zeta_{\mathbf{G}(\mathcal{O}_v)}(s)$ enumerates the continuous representations of the compact p -adic analytic group $\mathbf{G}(\mathcal{O}_v)$. Here, \mathcal{O}_v denotes the completion of \mathcal{O} at the place v of K . The Witten zeta functions are comparatively well understood; see, for instance, [15, Theorem 5.1] for their abscissa of convergence. To control the non-archimedean factors and the analytic properties of their Euler product seems much harder. The factors are not, in general, rational functions in p^{-s} , as the groups $\mathbf{G}(\mathcal{O}_v)$ are only virtually pro- p . Consequently, formulae of Denef type (cf. (1.4)) cannot be expected for these Euler factors in general. A result by Jaikin ([13]) expresses the zeta functions of the groups $\mathbf{G}(\mathcal{O}_v)$ as rational functions in p^{-s} and terms of the form n_i^{-s} for finitely many integers n_i , but to what degree these expressions may be described uniformly as the numbers of rational points of globally defined geometric objects (such as varieties or, more generally, definable sets) remains unclear.

Most general results about the analytic properties of Euler products of the non-archimedean factors are based on suitable approximations of the individual Euler factors. Using such techniques, it was shown that the abscissa of convergence of $\zeta_{\Gamma}(s)$ is a rational number ([2]) which only depends on the absolute root system of the algebraic group \mathbf{G} ; cf. [4]. For groups of type A_2 this abscissa is equal to 1, the zeta function allows for meromorphic continuation to at least $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > 5/6\}$, and the continued function is analytic in this domain with the exception of a double pole at $s = 1$; cf. [5]. It is not known whether this kind of uniform analytic properties are a general feature of the arithmetic groups considered in this context. In [1], Aizenbud and Avni establish absolute upper bounds for the abscissae of convergence of representation zeta functions of arithmetic lattices of higher rank. We refer to [14] and [28] for surveys of representation zeta functions of groups.

1.3.3. (Normal) subgroup growth of nilpotent groups. One precursor of and source of inspiration in the study of zeta functions in representation growth is the use of these functions in the related area of subgroup growth of groups. The idea to employ zeta functions to enumerate finite-index subgroups of finitely generated nilpotent groups was pioneered in [11] by Grunewald, Segal, and Smith and subsequently developed by many authors; see [18, 27] and references therein. Zeta functions enumerating finite-index (normal) subgroups of finitely generated nilpotent groups are Dirichlet generating functions which satisfy decompositions akin to (1.2) as Euler products of rational functions, indexed by the rational primes. In [8], du Sautoy and Grunewald produced formulae of Denef type for the Euler factors arising in this context, by expressing almost all of them as p -adic *cone integrals*. One of the main results of their paper establishes analytic properties of Euler products of cone integrals; cf. [8, Theorem 1.5]. Together with some control over remaining Euler factors, this allows them to establish [8, Theorem 5.7], an analogue of our Corollary B for (normal) subgroup zeta functions of groups.

The analogy notwithstanding, Corollary B does not seem to follow directly from the results in [8], as it is not known whether (almost all of) the Euler factors of representation zeta functions of finitely generated nilpotent groups are cone integrals. This is one of the reasons why novel p -adic methods for the description of the relevant local zeta functions were introduced in [12] and [26]. The latter paper produced explicit formulae of Denef type for almost all Euler factors in (1.2). The ideas behind this analysis were refined in [25] to produce such formulae for (almost all of) the factors of the “fine” Euler products (1.3). In the current paper, we adapt ideas pioneered in [8] for the study of Euler products of such formulae, regardless of the question – which we cannot decide – whether or not they are cone integrals.

Having commented on some similarities between subgroup growth and representation growth of finitely generated nilpotent groups, we mention a striking dissimilarity: uniform results such as Theorem A, establishing the independence of key analytic data of zeta functions under base extension, do not hold for normal zeta functions

of finitely generated nilpotent groups. This is already witnessed by the zeta function $\zeta_{\mathbf{H}(\mathcal{O})}^{\triangleleft}(s)$ enumerating finite-index normal subgroups of the Heisenberg groups $\mathbf{H}(\mathcal{O})$ over \mathcal{O} : it satisfies a “coarse” Euler product similar to (1.2), but not a “fine” one indexed by the places of L , such as (1.3). The (coarse) Euler factors depend in a combinatorially intricate manner on the indexing prime’s decomposition in L ; cf. [11, Theorem 3]. The abscissa of convergence of $\zeta_{\mathbf{H}(\mathcal{O})}^{\triangleleft}(s)$ is $2[L : \mathbb{Q}]$; see [22, 21] for further details.

1.4. Organization and notation. In Section 2, we recall from [25] some tools for the analysis of local representation zeta functions arising as factors in products such as (1.3) in terms of Poincaré series associated to matrices of linear forms and \mathfrak{p} -adic integrals. In Section 3, we combine results of and techniques developed in [8], [25], and [3] to represent these local zeta functions by formulae of Denef type. In particular, we prove that the abscissae of convergence of the local zeta functions are all elements of a finite set of rational numbers (cf. Corollary 3.9) which is strictly dominated by the (global) abscissa of convergence. Theorem A and Corollary B are proved in Sections 4 and 5, respectively.

Given a ring A , we write $\text{Mat}_{n \times m}(A)$ for the set of all $n \times m$ matrices with coefficients in A and $\text{Mat}_n(A)$ for $\text{Mat}_{n \times n}(A)$. We write \mathbb{N} for the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Given $k \in \mathbb{N}_0$, we set $[k] = \{1, \dots, k\}$ and $[k]_0 = [k] \cup \{0\}$. Given $r \in \mathbb{R}$, we write $\lfloor r \rfloor$ for the largest integer not exceeding r . Given a variable x and a set of polynomials $F(y)$, we write $xF(y)$ for the set $\{xf(y) \mid f(y) \in F(y)\}$.

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2. REPRESENTATION ZETA FUNCTIONS OF FINITELY GENERATED NILPOTENT GROUPS

Let K be a number field and \mathcal{O}_K its ring of integers. Let Λ be a nilpotent \mathcal{O}_K -Lie lattice of nilpotency class c . Without loss of generality $c \geq 2$. If $c > 2$ we assume that $\Lambda' \subseteq c!\Lambda$. Let $\mathbf{G} = \mathbf{G}_{\Lambda}$ be the unipotent group scheme over \mathcal{O}_K associated to Λ as in [25, Sections 2.1.2 and 2.4.1]. Let L be a finite extension of K and denote by $\mathcal{O} = \mathcal{O}_L$ its ring of integers. For a nonzero prime ideal \mathfrak{p} of \mathcal{O} , we denote by $\mathcal{O}_{\mathfrak{p}}$ the completion of \mathcal{O} at \mathfrak{p} , with maximal ideal \mathfrak{p} , residue field cardinality q , and residue field characteristic p . Recall that the Euler product (1.3) relates the zeta function of the \mathcal{T} -group $\mathbf{G}(\mathcal{O})$ to the zeta functions of the nilpotent pro- p groups $\mathbf{G}(\mathcal{O}_{\mathfrak{p}})$, enumerating the continuous twist isoclasses of continuous irreducible complex representations of $\mathbf{G}(\mathcal{O}_{\mathfrak{p}})$.

In this section, we recapitulate some notation and constructions from [25] to express almost all of the Euler factors $\zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s)$ in (1.3) in terms of Poincaré series associated

to matrices of linear forms and these series, in turn, in terms of \mathfrak{p} -adic integrals. The key tool for this approach is the Kirillov orbit method for pro- p groups such as $\mathbf{G}(\mathcal{O}_{\mathfrak{p}})$. When it is applicable, this method allows us to translate the problem of counting irreducible representations into the problem of counting co-adjoint orbits of an associated Lie algebra. It turns out that, for all but finitely many prime ideals \mathfrak{p} , the Kirillov orbit method applies to $\mathbf{G}(\mathcal{O}_{\mathfrak{p}})$; in the remaining cases it applies to suitable finite index subgroups of $\mathbf{G}(\mathcal{O}_{\mathfrak{p}})$.

In Section 3 we will use the results of the current section to develop formulae of Denef type which uniformly describe almost all of the Euler factors of the product (1.3). They will allow us, in particular, to define a finite set \mathbf{P} of rational numbers, depending only on \mathbf{G} , containing the abscissae of convergence of *all* the Euler factors $\zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s)$. (Here, the abscissa of convergence of the zeta function of a finitely generated nilpotent pro- p group is defined as for abstract finitely generated nilpotent groups.) In this context, we will refer to the following elementary lemma which implies that the abscissa of convergence of the zeta function of a finitely generated nilpotent pro- p group is a commensurability invariant. Its proof is analogous to that of [24, Proposition 4.13], which in turn is a modification of that of [17, Lemma 2.2].

Lemma 2.1. *Let p be a prime, G a finitely generated nilpotent pro- p group, and $H \leq G$ a subgroup of finite index in G . Write $|G : H| = p^a$ and $|G' \cap H : H'| = p^b$ for $a, b \in \mathbb{N}_0$. For $n \in \mathbb{N}_0$, let $\tilde{R}_{p^n}(G) = \sum_{i=0}^n \tilde{r}_{p^i}(G)$ and $\tilde{R}_{p^n}(H) = \sum_{i=0}^n \tilde{r}_{p^i}(H)$ denote the numbers of continuous twist isoclasses of complex irreducible representations of dimension at most p^n of G and H , respectively. Then*

$$\tilde{R}_{p^n}(H) \leq p^a \tilde{R}_{p^{a+n}}(G) \quad \text{and} \quad \tilde{R}_{p^n}(G) \leq p^{a+b} \tilde{R}_{p^n}(H).$$

In particular, the abscissa of convergence of the representation zeta function $\zeta_G(s)$ is a commensurability invariant of G , i.e. $\alpha(G) = \alpha(H)$.

Proof. For $i \in \mathbb{N}_0$, let $\widetilde{\text{Irr}}_{p^i}(G)$ resp. $\widetilde{\text{Irr}}_{p^i}(H)$ be the set of twist isoclasses of G resp. H of dimension p^i . (Here and in the sequel, the dimension of a twist isoclass denotes, of course, the common dimension of all of its elements.) For $i \in [n]_0$ and $\tilde{\sigma} \in \widetilde{\text{Irr}}_{p^i}(H)$, let $\tilde{\Phi}(\tilde{\sigma})$ be the twist isoclass of an irreducible constituent of $\text{Ind}_H^G \sigma$, where σ is any representative of the twist isoclass $\tilde{\sigma}$. This defines a (noncanonical) map

$$\tilde{\Phi} : \bigcup_{i=0}^n \widetilde{\text{Irr}}_{p^i}(H) \rightarrow \bigcup_{i=0}^{n+a} \widetilde{\text{Irr}}_{p^i}(G).$$

To establish the first inequality it suffices to show that the fibre sizes of $\tilde{\Phi}$ are all bounded by p^a . Fix thus $\tilde{\rho} \in \bigcup_{i=0}^{n+a} \widetilde{\text{Irr}}_{p^i}(G)$ and consider $\tilde{\Phi}^{-1}(\tilde{\rho}) = \{\tilde{\sigma}_1, \dots, \tilde{\sigma}_m\} \subseteq \bigcup_{i=0}^n \widetilde{\text{Irr}}_{p^i}(H)$. Without loss of generality we may assume that $\dim \tilde{\sigma}_1 \leq \dim \tilde{\sigma}_i$ for all $i \in \{2, \dots, m\}$.

By definition of $\tilde{\Phi}$, there exist $\sigma_i \in \tilde{\sigma}_i$ and $\rho_i \in \tilde{\rho}$ such that $\rho_i \in \text{Ind}_H^G \sigma_i$ for each i . By Frobenius Reciprocity, $\sigma_i \in \text{Res}_H^G \rho_i$ for each i . There exist thus continuous 1-dimensional representations χ_i of G such that $(\text{Res}_H^G \chi_i) \otimes \sigma_i \in \text{Res}_H^G \rho_1$. This implies

that $\text{Res}_H^G \rho_1 = (\text{Res}_H^G \chi_1) \otimes \sigma_1 \oplus \cdots \oplus (\text{Res}_H^G \chi_m) \otimes \sigma_m \oplus \rho'$ for some continuous representation ρ' of H . Hence $m \dim \sigma_1 \leq \dim \rho_1 \leq p^a \dim \sigma_1$ and thus $m \leq p^a$.

The proof of the second inequality is similar. \square

2.1. Poincaré series. We now fix a nonzero prime ideal \mathfrak{p} of \mathcal{O} and write $\mathfrak{o} = \mathcal{O}_{\mathfrak{p}}$. Let $\mathfrak{g} := \Lambda(\mathfrak{o}) = (\Lambda \otimes_{\mathcal{O}_K} \mathcal{O}) \otimes_{\mathcal{O}} \mathfrak{o}$ and write \mathfrak{z} for the centre of \mathfrak{g} . Set $h = \text{rk}_{\mathfrak{o}}(\mathfrak{g})$ and

$$d = \text{rk}_{\mathfrak{o}}(\mathfrak{g}'), \quad k = \text{rk}_{\mathfrak{o}}(\iota(\mathfrak{g}')/\iota(\mathfrak{g}' \cap \mathfrak{z})) = \text{rk}_{\mathfrak{o}}(\iota(\mathfrak{g}' + \mathfrak{z})/\mathfrak{z}), \quad r - k = \text{rk}_{\mathfrak{o}}(\mathfrak{g}/\iota(\mathfrak{g}' + \mathfrak{z})),$$

so that $r = \text{rk}_{\mathfrak{o}}(\mathfrak{g}/\mathfrak{z})$. Here, for a finitely generated R -module M with R either \mathcal{O} or \mathfrak{o} and an R -submodule N of M , we denote by $\iota(N)$ the isolator of N in M , that is the smallest submodule L of M containing N such that M/L is torsion-free. Note that \mathfrak{z} is isolated in \mathfrak{g} , i.e. $\iota(\mathfrak{z}) = \mathfrak{z}$; see [25, Lemma 2.5]. Notice also that $c \leq 2$ if and only if $k = 0$.

We choose a uniformizer π of \mathfrak{o} , write $-$ for the natural surjection $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{z}$, and choose an (\mathfrak{o}) -basis $\mathbf{e} = (e_1, \dots, e_h)$ for \mathfrak{g} such that

$$\begin{aligned} (e_{r-k+1}, \dots, e_r) &\text{ is a basis for } \iota(\overline{\mathfrak{g}' + \mathfrak{z}}), \\ (e_{r+1}, \dots, e_{r-k+d}) &\text{ is a basis for } \iota(\mathfrak{g}' \cap \mathfrak{z}), \text{ and} \\ (e_{r+1}, \dots, e_h) &\text{ is a basis for } \mathfrak{z}. \end{aligned}$$

It follows from the elementary divisor theorem that there exist $b_1, \dots, b_d \in \mathbb{N}_0$ such that $(\overline{\pi^{b_1} e_{r-k+1}}, \dots, \overline{\pi^{b_k} e_r})$ and $(\pi^{b_{k+1}} e_{r+1}, \dots, \pi^{b_d} e_{r-k+d})$ are bases for $\overline{\mathfrak{g}' + \mathfrak{z}}$ and $\mathfrak{g}' \cap \mathfrak{z}$ respectively. We choose a basis $\mathbf{f} = (f_1, \dots, f_d)$ for \mathfrak{g}' such that

$$\begin{aligned} (\overline{f_1}, \dots, \overline{f_k}) &= (\overline{\pi^{b_1} e_{r-k+1}}, \dots, \overline{\pi^{b_k} e_r}) \text{ and} \\ (f_{k+1}, \dots, f_d) &= (\pi^{b_{k+1}} e_{r+1}, \dots, \pi^{b_d} e_{r-k+d}); \end{aligned}$$

cf. [25, Section 2.2.2].

For $i, j \in [r]$ and $l \in [d]$, let $\lambda_{ij}^l \in \mathfrak{o}$ be the structure constants with respect to above bases, i.e. $[e_i, e_j] = \sum_{l=1}^d \lambda_{ij}^l f_l$. We define the commutator matrix

$$\mathcal{R}(\mathbf{Y}) = \left(\sum_{l=1}^d \lambda_{ij}^l Y_l \right)_{ij} \in \text{Mat}_r(\mathfrak{o}[\mathbf{Y}])$$

of \mathfrak{g} with respect to the chosen bases and let $\mathcal{S}(\mathbf{Y})$ be the $r \times k$ submatrix of $\mathcal{R}(\mathbf{Y})$ comprising the last k columns of $\mathcal{R}(\mathbf{Y})$. If $c = 2$, then the matrix \mathcal{S} does not feature since $k = 0$.

Let $m \in \mathbb{N}_0$ with $m \leq r$. We say that a matrix $S \in \text{Mat}_{r \times m}(\mathfrak{o})$ has (elementary divisor) type $\tilde{\nu}(S) = \mathbf{c} = (c_1, \dots, c_m) \in (\mathbb{N}_0 \cup \{\infty\})^m$ if S is equivalent to

$$\begin{pmatrix} \pi^{c_1} & & \\ & \ddots & \\ & & \pi^{c_m} \end{pmatrix} \in \text{Mat}_{r \times m}(\mathfrak{o}),$$

with $0 \leq c_1 \leq \dots \leq c_m$. Let $R \in \text{Mat}_r(\mathfrak{o})$ be an antisymmetric matrix. We set $\nu(R) = (a_1, \dots, a_{\lfloor r/2 \rfloor}) \in (\mathbb{N}_0 \cup \{\infty\})^{\lfloor r/2 \rfloor}$, where $0 \leq a_1 \leq \dots \leq a_{\lfloor r/2 \rfloor}$, if

$$\tilde{\nu}(R) = \begin{cases} (a_1, a_1, a_2, a_2, \dots, a_{r/2, r/2}) & \text{if } r \text{ is even,} \\ (a_1, a_1, a_2, a_2, \dots, a_{(r-1)/2, (r-1)/2}, \infty) & \text{if } r \text{ is odd.} \end{cases}$$

Let $N \in \mathbb{N}_0$. Given an antisymmetric matrix $\bar{R} \in \text{Mat}_r(\mathfrak{o}/\mathfrak{p}^N)$, we set $\nu(\bar{R}) := (\min\{a_i, N\})_{i \in [\lfloor r/2 \rfloor]} \in ([N]_0)^{\lfloor r/2 \rfloor}$, where $\mathbf{a} = (a_1, \dots, a_{\lfloor r/2 \rfloor}) = \nu(R)$ is the type of any lift R of \bar{R} under the natural surjection $\text{Mat}_r(\mathfrak{o}) \rightarrow \text{Mat}_r(\mathfrak{o}/\mathfrak{p}^N)$. Given $\bar{S} \in \text{Mat}_{r \times k}(\mathfrak{o}/\mathfrak{p}^N)$, the vector $\tilde{\nu}(\bar{S}) \in ([N]_0)^k$ is defined similarly. We set $W_N(\mathfrak{o}) := (\mathfrak{o}/\mathfrak{p}^N)^d \setminus (\mathfrak{p}/\mathfrak{p}^N)^d$ if $N \in \mathbb{N}$ and $W_0(\mathfrak{o}) = \{0\}$. Given $N \in \mathbb{N}_0$, $\mathbf{a} \in \mathbb{N}_0^{\lfloor r/2 \rfloor}$, and $\mathbf{c} \in \mathbb{N}_0^k$, we set

$$\mathcal{N}_{N, \mathbf{a}, \mathbf{c}}^{\mathfrak{o}} := \#\{\mathbf{y} \in W_N(\mathfrak{o}) \mid \nu(\mathcal{R}(\mathbf{y})) = \mathbf{a}, \tilde{\nu}(\mathcal{S}(\mathbf{y}) \cdot \text{diag}(\pi^{b_1}, \dots, \pi^{b_k})) = \mathbf{c}\},$$

giving rise to the Poincaré series

$$(2.1) \quad \mathcal{P}_{\mathcal{R}, \mathcal{S}, \mathfrak{o}}(s) := \sum_{\substack{N \in \mathbb{N}_0 \\ \mathbf{a} \in \mathbb{N}_0^{\lfloor r/2 \rfloor}, \mathbf{c} \in \mathbb{N}_0^k}} \mathcal{N}_{N, \mathbf{a}, \mathbf{c}}^{\mathfrak{o}} q^{-\sum_{i=1}^{\lfloor r/2 \rfloor} (N - a_i)s - \sum_{i=1}^k (N - c_i)}.$$

Proposition 2.2. [25, Propositions 2.9 and 2.18] *If $(p, c) \neq (2, 3)$, then*

$$\zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s) = \mathcal{P}_{\mathcal{R}, \mathcal{S}, \mathcal{O}_{\mathfrak{p}}}(s).$$

Key to writing the zeta function $\zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s)$ as a Poincaré series is the fact that the hypothesis of Proposition 2.2 ensures that the Kirillov orbit method is applicable to the pro- p group $\mathbf{G}(\mathcal{O}_{\mathfrak{p}})$; see [25, Section 2.2] for details.

Assume now that $p = 2$ and $c = 3$. In this case we consider, instead of \mathfrak{g} , a suitable congruence sublattice of \mathfrak{g} giving rise to a pro-2 group to which the Kirillov orbit method is applicable. Let $e = e(\mathfrak{o}, \mathbb{Z}_2)$ be the absolute ramification index of \mathfrak{o} , i.e. $2\mathfrak{o} = \mathfrak{p}^e \mathfrak{o}$, and consider $\mathfrak{g}^e = 2\Lambda \otimes_{\mathcal{O}} \mathfrak{o}$. Since $[2\Lambda, 2\Lambda] = 4\Lambda' \subseteq 4 \cdot 3! \Lambda \subseteq 4 \cdot 2\Lambda$, we have that $(\mathfrak{g}^e)' \subseteq 4\mathfrak{g}^e$, i.e. \mathfrak{g}^e is powerful. Hence the group $\mathbf{G}^e(\mathfrak{o}) := \exp(\mathfrak{g}^e)$ is a powerful pro-2 group. Moreover, since $\mathbf{G}^e(\mathfrak{o})$ is finitely generated torsion-free, [7, Theorem 4.5] yields that the group $\mathbf{G}^e(\mathfrak{o})$ is a uniform pro-2 group. By [13, Theorem 2.12], there exists a Kirillov correspondence between the finite co-adjoint orbits in the dual of the Lie algebra \mathfrak{g}^e and the continuous irreducible representations of $\mathbf{G}^e(\mathfrak{o})$. Multiplying the chosen basis \mathbf{e} for \mathfrak{g} by π^e yields a basis $\pi^e \mathbf{e}$ for \mathfrak{g}^e . Hence, the zeta function $\zeta_{\mathbf{G}^e(\mathfrak{o})}(s)$ can be expressed in term of a Poincaré series as in (2.1).

2.2. \mathfrak{p} -Adic integration. It follows from [26, Section 2.2] that

$$(2.2) \quad \mathcal{P}_{\mathcal{R}, \mathcal{S}, \mathfrak{o}}(s) = 1 + (1 - q^{-1})^{-1} \mathcal{Z}_{\mathfrak{o}}(-s/2, -1, us + v - d - 1),$$

where $\mathcal{Z}_{\mathfrak{o}}(\rho, \sigma, \tau)$ is the \mathfrak{p} -adic integral defined in [25, (2.8)] as follows:

$$(2.3) \quad \mathcal{Z}_{\mathfrak{o}}(\rho, \sigma, \tau) = \int_{(x, \mathbf{y}) \in \mathfrak{p} \times (\mathfrak{o}^d)^*} |x|_{\mathfrak{p}}^{\tau} \prod_{j=1}^u \frac{\|F_j(\mathbf{y}) \cup F_{j-1}(\mathbf{y})x^2\|_{\mathfrak{p}}^{\rho}}{\|F_{j-1}(\mathbf{y})\|_{\mathfrak{p}}^{\rho}} \prod_{i=1}^v \frac{\|G_i(\mathbf{y}) \cup G_{i-1}(\mathbf{y})x\|_{\mathfrak{p}}^{\sigma}}{\|G_{i-1}(\mathbf{y})\|_{\mathfrak{p}}^{\sigma}} d\mu(x, \mathbf{y}).$$

Here μ is the additive Haar measure on \mathfrak{o}^{d+1} is normalised so that $\mu(\mathfrak{o}^{d+1}) = 1$, and

$$(2.4) \quad \begin{aligned} 2u &= \max\{\mathrm{rk}_{\mathrm{Frac}(\mathfrak{o})}(\mathcal{R}(\mathbf{z})) \mid \mathbf{z} \in \mathfrak{o}^d\}, \\ v &= \max\{\mathrm{rk}_{\mathrm{Frac}(\mathfrak{o})}(\mathcal{S}(\mathbf{z})) \mid \mathbf{z} \in \mathfrak{o}^d\}, \\ F_j(\mathbf{Y}) &= \{f \mid f = f(\mathbf{Y}) \text{ a principal } 2j \times 2j \text{ minor of } \mathcal{R}(\mathbf{Y})\}, \\ G_i(\mathbf{Y}) &= \{g \mid g = g(\mathbf{Y}) \text{ an } i \times i \text{ minor of } \mathcal{S}(\mathbf{Y}) \cdot \mathrm{diag}(\pi^{b_1}, \dots, \pi^{b_k})\}, \\ \|H(X, \mathbf{Y})\|_{\mathfrak{p}} &= \max\{|h(X, \mathbf{Y})|_{\mathfrak{p}} \mid h \in H\} \text{ for a finite set } H \subset \mathfrak{o}[X, \mathbf{Y}]. \end{aligned}$$

Notice that $v = 0$ if $c = 2$.

3. FORMULAE OF DENEUF TYPE FOR LOCAL REPRESENTATION ZETA FUNCTIONS

In this section we produce uniform formulae of Denef type for (almost all of the) local zeta functions $\zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s)$ arising as Euler factors in (1.3). They enable us to obtain information about the abscissae of convergence of all Euler factors (Corollary 3.9) and will be put to use in Section 4, where Theorem A will be proven.

For a nonzero prime ideal \mathfrak{p} of \mathcal{O} , we write $\mathfrak{o} = \mathcal{O}_{\mathfrak{p}}$ for the completion of \mathcal{O} at \mathfrak{p} . As we have seen in Section 2, almost all of the factors $\zeta_{\mathbf{G}(\mathfrak{o})}(s)$ in the Euler product (1.3) are given by Poincaré series $\mathcal{P}_{\mathcal{R}, \mathcal{S}, \mathfrak{o}}(s)$, which can be expressed in terms of the p -adic integral $\mathcal{Z}_{\mathfrak{o}}(\rho, \sigma, \tau)$ defined in (2.3). In order to analyze such integrals in a uniform manner we recall from [26, Section 2] some formulae for \mathfrak{p} -adic integrals of the following general form. Fix $d, l \in \mathbb{N}$. Let $I_{\kappa}, \kappa \in [l]$, be finite index sets, x and $\mathbf{y} = (y_{ij})_{1 \leq i, j \leq d}$ integration variables, and fix nonnegative integers $e_{\kappa\iota}$ and finite sets of polynomials $F_{\kappa\iota}(\mathbf{Y})$ over \mathcal{O}_K for $\kappa \in [l], \iota \in I_{\kappa}$. We define

$$(3.1) \quad Z(\mathbf{s}) = Z(s_1, \dots, s_l) := \int_{(x, \mathbf{y}) \in \mathfrak{p} \times \mathrm{GL}_d(\mathfrak{o})} \prod_{\kappa \in [l]} \left\| \bigcup_{\iota \in I_{\kappa}} x^{e_{\kappa\iota}} F_{\kappa\iota}(\mathbf{y}) \right\|_{\mathfrak{p}}^{s_{\kappa}} d\mu(x, \mathbf{y}).$$

Here $\mathbf{s} = (s_1, \dots, s_l)$ is a vector of complex variables and $d\mu$ denotes the product of the additive Haar measure on \mathfrak{p} and the Haar measure on $\mathrm{GL}_d(\mathfrak{o})$, normalized so that $d\mu(\mathfrak{p} \times \mathrm{GL}_d(\mathfrak{o})) = q^{-1} \prod_{i=1}^d (1 - q^{-i})$. We assume throughout that the ideals $(F_{\kappa\iota})$ are all invariant under the natural (right-)action of the standard Borel subgroup $B \subseteq \mathrm{GL}_d$. Equation (3.1) is a slight specialization of the integral given in [26, eq. (6)]: we make some of the hypotheses of [26, Theorem 2.2] and restrict to the case $I = \{1\}$. Note that whilst (3.1) is a \mathfrak{p} -adic (“local”) integral, its integrand is defined over \mathcal{O}_K , i.e. “globally”.

We fix a principalization (Y, h) with $h : Y \rightarrow \mathrm{GL}_d/B$ of the \mathcal{O}_K -ideal

$$\mathcal{I} = \prod_{\kappa \in [l]} \prod_{\iota \in I_\kappa} (F_{\kappa\iota}).$$

Let $(N_{u\kappa\iota}, \nu_u)_{u\kappa\iota}$ be the associated numerical data, with $u \in T$, $\kappa \in [l]$, and $\iota \in I_\kappa$. Here, T is a finite set indexing the irreducible components E_u of the pre-image under h of the variety defined by \mathcal{I} . For u, κ, ι as above, the number $\nu_u - 1$ denotes the multiplicity of E_u in the divisor $h^*(d\mu(\mathbf{y}))$ and $N_{u\kappa\iota}$ denotes the multiplicity of E_u in the pre-image under h of the variety defined by the ideal $(F_{\kappa\iota})$. The principalization (Y, h) has good reduction modulo \mathfrak{p} (cf. [6, Definition 2.2]) for all but finitely many \mathfrak{p} ; cf. [6, Theorem 2.4]. In the remaining cases we say that (Y, h) has bad reduction modulo \mathfrak{p} .

In Sections 3.1 and 3.2 we compute formulae of Denef type for $Z(\mathbf{s})$ in case that (Y, h) has good respectively bad reduction modulo \mathfrak{p} . In Section 3.3 we specialize these formulae to the integrals $\mathcal{Z}_o(\rho, \sigma, \tau)$ defined in Section 2.2.

3.1. Good reduction. Assume that \mathfrak{p} is such that (Y, h) has good reduction modulo \mathfrak{p} . According to [26, Theorem 2.2],

$$(3.2) \quad Z(\mathbf{s}) = \frac{(1 - q^{-1})^{d+1}}{q^{\binom{d}{2}}} \sum_{U \subseteq T} c_U(\mathfrak{o}/\mathfrak{p}) (q - 1)^{|U|} \Xi_U(q, \mathbf{s}).$$

Here, for each $U \subseteq T$, the coefficient $c_U(\mathfrak{o}/\mathfrak{p})$ is the number of $\mathfrak{o}/\mathfrak{p}$ -rational points of $\overline{E_U} \setminus \cup_{V \supsetneq U} \overline{E_V}$ ($E_U := \cap_{u \in U} E_u$), where $\overline{}$ denotes reduction modulo \mathfrak{p} . Furthermore, q denotes the cardinality of the residue field $\mathfrak{o}/\mathfrak{p}$ and

$$(3.3) \quad \Xi_U(q, \mathbf{s}) = \sum_{\substack{(m_u)_{u \in U} \in \mathbb{N}^{|U|} \\ m_{t+1} \in \mathbb{N}}} q^{-\mathcal{L}((m_u)_{u \in U}, m_{t+1}) - \sum_{\kappa \in [l]} s_\kappa \min\{\mathcal{L}_{\kappa\iota}((m_u)_{u \in U}, m_{t+1}) \mid \iota \in I_\kappa\}},$$

where $t = |T|$ and

$$(3.4) \quad \begin{aligned} \mathcal{L}((m_u)_{u \in U}, m_{t+1}) &= m_{t+1} + \sum_{u \in U} \nu_u m_u, \\ \mathcal{L}_{\kappa\iota}((m_u)_{u \in U}, m_{t+1}) &= e_{\kappa\iota} m_{t+1} + \sum_{u \in U} N_{u\kappa\iota} m_u, \text{ for } \kappa \in [l], \iota \in I_\kappa. \end{aligned}$$

Remark 3.1. We denote here by $c_U(\mathfrak{o}/\mathfrak{p})$ what is called $c_U(q)$ in [25] and [26]. The latter notation obscures the fact that this quantity may depend on the prime ideal \mathfrak{p} and not just on the cardinality of the residue field. (Texts such as [6] avoided this pitfall.) We are grateful to Tobias Rossmann for pointing this out to us.

We uniformly rewrite the functions $\Xi_U(q, \mathbf{s})$ in terms of zeta functions of polyhedral cones in a fan, in close analogy to the proof of [8, Lemma 3.1]. We first extend the domain of summation in $\Xi_U(q, \mathbf{s})$ by taking the sum over all $\mathbf{m} = (m_1, \dots, m_{t+1}) \in \mathbb{N}_0^t \times \mathbb{N}$ with $m_u = 0$ iff $u \in T \setminus U$. Choose a finite triangulation $\{R_i\}_{i \in [w]_0}$ of $\mathbb{R}_{\geq 0}^{t+1}$ consisting of relatively open, pairwise disjoint *simple* rational polyhedral cones R_i , which eliminates the “min”-terms in the exponent of q in $\Xi_U(q, \mathbf{s})$. (It will be important for

future purposes to note that this decomposition can be chosen independently of \mathbf{p} , depending just on the chosen principalization (Y, h) . We assume that $R_0 = \{\mathbf{0}\}$ and R_1, \dots, R_z are exactly the one-dimensional cones (or rays) in $\{R_i\}_{i \in [w]}$. For every $j \in [z]$, write $R_j = \mathbb{R}_{>0} \mathbf{r}_j$, where $\mathbf{r}_j \in \mathbb{N}_0^{t+1}$ is the shortest integral vector on the ray R_j . For every $i \in [w]$ there exists a set $M_i \subseteq [z]$ of rays such that

$$R_i = \bigoplus_{j \in M_i} \mathbb{R}_{>0} \mathbf{r}_j$$

(direct sum of monoids). Note that $i \in [z]$ if and only if $|M_i| = 1$, in which case $M_i = \{i\}$. For $i \in [w]$, the simplicity of R_i yields that

$$(3.5) \quad R_i \cap \mathbb{N}_0^{t+1} = \bigoplus_{j \in M_i} \mathbb{N} \mathbf{r}_j.$$

For $U \subseteq T$, set

$$\mathcal{C}_U = \{\mathbf{m} \in \mathbb{N}_0^t \times \mathbb{N} \mid m_u = 0 \text{ iff } u \in T \setminus U\},$$

the domain of summation in (3.3). As $\bigcup_{U \subseteq T} \mathcal{C}_U = \mathbb{N}_0^t \times \mathbb{N}$, not all of the cones R_i , $i \in [w]$, are relevant for sums of the form (3.3). Rather, for each $U \subseteq T$ there exists a uniquely determined subset $W'_U \subseteq [w]$ such that

$$(3.6) \quad \mathcal{C}_U = \bigcup_{i \in W'_U} R_i \cap \mathbb{N}_0^{t+1}.$$

The set $W' := \bigcup_{U \subseteq T} W'_U \subseteq [w]$ indexes the cones which do not lie in the boundary component $\mathbb{R}_{\geq 0}^t \times \{0\}$ of $\mathbb{R}_{\geq 0}^{t+1}$.

Let $i \in W'$. Restricting the summation in (3.3) to R_i – or, indeed, its closure –, allows us to rewrite the exponent of q in this sum as

$$-\sum_{u=1}^{t+1} \left(\sum_{\kappa \in [l]} C_{u\kappa} s_\kappa + D_u \right) m_u$$

for suitable $C_{u\kappa}, D_u \in \mathbb{N}_{\geq 0}$ for all $u \in [t+1], \kappa \in [l]$. For $j \in [z]$, write

$$\mathbf{r}_j = (r_{j1}, \dots, r_{jt+1}) \in \mathbb{N}_{\geq 0}^{t+1}.$$

For each $j \in M_i$ and $\kappa \in [l]$, set

$$A_{j\kappa} = \sum_{u=1}^{t+1} r_{ju} C_{u\kappa} \quad \text{and} \quad B_j = \sum_{u=1}^{t+1} r_{ju} D_u.$$

Note that $B_j > 0$ for all j as, for $u \in [t]$, we find that $r_{ju} > 0$ iff $D_u > 0$. Combining (3.6) and (3.5) yields, for $U \subseteq T$,

$$(3.7) \quad \begin{aligned} \Xi_U(q, \mathbf{s}) &= \sum_{i \in W'_U} \sum_{\mathbf{m} \in R_i \cap (\mathbb{N}_0^t \times \mathbb{N})} q^{-\sum_{u=1}^{t+1} (\sum_{\kappa \in [l]} C_{u\kappa} s_\kappa + D_u) m_u} \\ &= \sum_{i \in W'_U} \prod_{j \in M_i} \frac{q^{-(\sum_{\kappa \in [l]} A_{j\kappa} s_\kappa + B_j)}}{1 - q^{-(\sum_{\kappa \in [l]} A_{j\kappa} s_\kappa + B_j)}}. \end{aligned}$$

Given $i \in W'$, set

$$(3.8) \quad c_i(\mathfrak{o}/\mathfrak{p}) = c_U(\mathfrak{o}/\mathfrak{p}) \text{ and } U_i = U,$$

where $U \subseteq T$ is the unique subset of T such that $i \in W'_U$.

Proposition 3.2. *Assume that (Y, h) has good reduction modulo \mathfrak{p} . Then*

$$(3.9) \quad Z(\mathbf{s}) = \frac{(1 - q^{-1})^{d+1}}{q^{\binom{d}{2}}} \sum_{i \in W'} c_i(\mathfrak{o}/\mathfrak{p}) (q - 1)^{|U_i|} \prod_{j \in M_i} \frac{q^{-(\sum_{\kappa \in [l]} A_{j\kappa} s_\kappa + B_j)}}{1 - q^{-(\sum_{\kappa \in [l]} A_{j\kappa} s_\kappa + B_j)}}.$$

Proof. Observing that

$$\bigcup_{U \subseteq T} \bigcup_{i \in W'_U} R_i = \bigcup_{i \in W'} R_i = \mathbb{R}_{\geq 0}^t \times \mathbb{R}_{> 0}$$

and using (3.2) and (3.7), we obtain

$$\begin{aligned} Z(\mathbf{s}) &= \frac{(1 - q^{-1})^{d+1}}{q^{\binom{d}{2}}} \sum_{U \subseteq T} c_U(\mathfrak{o}/\mathfrak{p}) (q - 1)^{|U|} \sum_{i \in W'_U} \prod_{j \in M_i} \frac{q^{-(\sum_{\kappa \in [l]} A_{j\kappa} s_\kappa + B_j)}}{1 - q^{-(\sum_{\kappa \in [l]} A_{j\kappa} s_\kappa + B_j)}} \\ &= \frac{(1 - q^{-1})^{d+1}}{q^{\binom{d}{2}}} \sum_{i \in W'} c_i(\mathfrak{o}/\mathfrak{p}) (q - 1)^{|U_i|} \prod_{j \in M_i} \frac{q^{-(\sum_{\kappa \in [l]} A_{j\kappa} s_\kappa + B_j)}}{1 - q^{-(\sum_{\kappa \in [l]} A_{j\kappa} s_\kappa + B_j)}}. \quad \square \end{aligned}$$

3.2. Bad reduction. Assume now that \mathfrak{p} is one of the finitely many prime ideals of \mathcal{O} such that the principalization (Y, h) does not have good reduction modulo \mathfrak{p} . In this case, formula (3.9) may not hold. Informally speaking, this is due to the breakdown of Hensel's lemma if the components E_u are singular or have nonnormal crossings modulo \mathfrak{p} . To a certain extent, this failure can be mitigated by analyzing the integral defining $Z(\mathbf{s})$ – or rather its pull-back under the principalization h – on cosets modulo \mathfrak{p}^N for some $N \in \mathbb{N}$; in the case of good reduction, $N = 1$ is sufficient. This motivates the following definition, generalizing the rational functions Ξ_U introduced in (3.3). For $N \in \mathbb{N}$, $U \subseteq T$, $(d_{\kappa\iota}) \in \mathbb{N}_0^{\prod_{\kappa \in [l]} I_\kappa}$, set

$$(3.10) \quad \Xi_{U, (d_{\kappa\iota})}^N(q, \mathbf{s}) = \sum_{\substack{(m_u)_{u \in U} \in \mathbb{N}_{\geq N}^{|U|}, \\ m_{t+1} \in \mathbb{N}}} q^{-\mathcal{L}((m_u)_{u \in U}, m_{t+1}) - \sum_{\kappa \in [l]} s_\kappa \min\{\mathcal{L}_{\kappa\iota}((m_u)_{u \in U}, m_{t+1}) - d_{\kappa\iota} \mid \iota \in I_\kappa\}}.$$

Here, \mathcal{L} and $\{\mathcal{L}_{\kappa\iota}\}_{\kappa \in [l], \iota \in I_\kappa}$ are the linear forms defined in (3.4). The following is analogous to and proven in the same way as [3, Corollary 4.2].

Proposition 3.3. *There exist $N \in \mathbb{N}$, finite sets $J \subset \mathbb{N}_0$ and $\Delta \subset \mathbb{N}_0^{\prod_{\kappa \in [l]} I_\kappa}$, all depending on \mathfrak{p} , such that*

$$(3.11) \quad Z(\mathbf{s}) = \frac{(1 - q^{-1})^{d+1}}{q^{\binom{d}{2}}} \sum_{\substack{U \subseteq T, j \in J \\ (d_{\kappa\iota}) \in \Delta}} c_{U, j, (d_{\kappa\iota})}(\mathfrak{o}/\mathfrak{p}) (q^N - q^{N-1})^{|U|} q^{-j} \Xi_{U, (d_{\kappa\iota})}^N(q, \mathbf{s}),$$

where each $c_{U, j, (d_{\kappa\iota})}(\mathfrak{o}/\mathfrak{p})$ is the number of $\mathfrak{o}/\mathfrak{p}$ -rational points of a certain variety over $\mathfrak{o}/\mathfrak{p} \cong \mathbb{F}_q$.

Notice that (3.11) generalizes (3.2), the formula for good reduction; cf. [3, Section 4.3].

3.3. Application to Poincaré series and local representation zeta functions.

In this section we argue that all but finitely many of the Euler factors $\zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s)$ in (1.3) are described by (univariable substitutions of) \mathfrak{p} -adic integrals $\mathcal{Z}_{\mathfrak{o}}(\rho, \sigma, \tau)$ of the form (2.3) which fit into the general form (3.1). Hence, the conclusions of the previous two sections are applicable, allowing for a uniform description of these Euler factors. In particular, we will show that the real parts of the poles of these local representation functions – and thus, in particular, their abscissae of convergence – are all elements of a finite set \mathbf{P} of rational numbers, depending on \mathbf{G} but independent of L . Moreover, we will show that \mathbf{P} also contains the abscissae of convergence of the finitely many “exceptional” Euler factors not covered by the generic treatment. This will leave us, in Section 4, free to concentrate on analytic properties of the co-finite Euler product over the “generic” Euler factors.

Note that the bases \mathbf{e} and \mathbf{f} defined in Section 2, and thus the matrices $\mathcal{R}(\mathbf{Y})$ and $\mathcal{S}(\mathbf{Y})$ and the data $\mathbf{b} = (b_1, \dots, b_d) \in \mathbb{N}_0^d$, are defined only locally. In order to give uniform formulae for local zeta functions we need, however, a global source of local bases. As we do not assume that \mathcal{O} is a principal ideal domain, we may not hope for a global analogue of the construction of the bases \mathbf{e} and \mathbf{f} . Instead, we first choose an \mathcal{O}_K -basis $\mathbf{f} = (e_{r-k+1}, \dots, e_{r-k+d})$ for a free finite-index \mathcal{O}_K -submodule of the isolator $\iota(\Lambda')$. By [25, Lemma 2.5], it can be extended to an \mathcal{O}_K -basis \mathbf{e} for a free finite-index \mathcal{O}_K -submodule M of Λ . Let p be a rational prime.

If p divides neither $|\Lambda : M|$ nor $|\iota(\Lambda') : \Lambda'|$, then, for every prime ideal \mathfrak{p} of \mathcal{O} lying above p , tensoring the \mathcal{O}_K -basis \mathbf{e} with $\mathcal{O}_{\mathfrak{p}}$ yields an $\mathcal{O}_{\mathfrak{p}}$ -basis for $\Lambda(\mathcal{O}_{\mathfrak{p}}) \cong M(\mathcal{O}_{\mathfrak{p}})$ as in Section 2, with $\mathbf{b} = (0, \dots, 0)$. (Note that the condition that p does not divide $|\iota(\Lambda') : \Lambda'|$ is missing in the discussion of [25, Section 2.3]. This omission has no bearing on the proof of [25, Theorem A], as it applies to only finitely many prime ideals.) If, in addition, p is odd, then the zeta functions of the pro- p groups $\mathbf{G}(\mathcal{O}_{\mathfrak{p}}) = \exp(\Lambda(\mathcal{O}_{\mathfrak{p}}))$ are given by a univariable substitution of the multivariable \mathfrak{p} -adic integral $\mathcal{Z}_{\mathcal{O}_{\mathfrak{p}}}(\rho, \sigma, \tau)$ given in (2.3); cf. Proposition 2.2 and (2.2). As explained in detail in [3, Section 4.1.3], this integral fits into the framework developed in the current section. The key equation is the following analogue of [3, eq. (4.8)], which asserts that, for suitable data in (3.1) and vectors $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^l$,

$$(3.12) \quad \mathcal{Z}_{\mathcal{O}_{\mathfrak{p}}}(-s/2, -1, us + v - d - 1) = \frac{1}{\prod_{i=1}^{d-1} (1 - q^{-i})} Z(\mathbf{a}\mathbf{s} + \mathbf{b}).$$

The fundamental idea is to extend the integral $\mathcal{Z}_{\mathcal{O}_{\mathfrak{p}}}$ trivially to an integral over $\mathfrak{p} \times \mathrm{GL}_d(\mathfrak{o})$, interpreting $(\mathfrak{o}^d)^*$ as the space of first columns, say, of matrices in $\mathrm{GL}_d(\mathfrak{o})$; cf. [26, Remark 2.1]. The postulated B -invariance holds because, for each i and each j , the polynomials in the sets $F_j(\mathbf{Y})$ and $G_i(\mathbf{Y})$ defined in (2.4) and (2.5) are all homogeneous of the same degree, depending only on i resp. j . The global nature of

the bases ensures that they are defined over \mathcal{O}_K . Hence the analysis of Sections 3.1 and 3.2 applies if p does not divide $|\Lambda : M|$ or $|\iota(\Lambda') : \Lambda'|$.

If p does divide $|\Lambda : M|$ or $|\iota(\Lambda') : \Lambda'|$, then $M(\mathcal{O}_{\mathfrak{p}})$ is a finite-index sublattice of $\Lambda(\mathcal{O}_{\mathfrak{p}})$ (proper if and only if p divides $|\Lambda : M|$). Whilst $M(\mathcal{O}_{\mathfrak{p}})$ may not even be a Lie lattice, there exists a constant $C = C(\mathfrak{p}) \in \mathbb{N}_0$ such that, for all $m \geq C$, the Lie lattice $\mathfrak{p}^m M(\mathcal{O}_{\mathfrak{p}})$ gives rise, by p -adic Lie theory, to a finitely generated nilpotent pro- p group $\mathbf{H}^m(\mathcal{O}_{\mathfrak{p}}) := \exp(\mathfrak{p}^m M(\mathcal{O}_{\mathfrak{p}}))$ to which the Kirillov orbit method is applicable. Indeed, it suffices to ensure that $\mathbf{H}^m(\mathcal{O}_{\mathfrak{p}})$ is uniformly powerful; cf. [13, Theorem 2.12]. We entertain no hope to describe the zeta functions of the groups $\mathbf{H}^m(\mathcal{O}_{\mathfrak{p}})$ explicitly in the given generality, but can still make statements about their abscissae of convergence. Note that, as all of the groups $\mathbf{H}^m(\mathcal{O}_{\mathfrak{p}})$ have finite index in $\mathbf{G}(\mathcal{O}_{\mathfrak{p}})$, the abscissae of convergence $\alpha(\mathbf{H}^m(\mathcal{O}_{\mathfrak{p}}))$ and $\alpha(\mathbf{G}(\mathcal{O}_{\mathfrak{p}}))$ all coincide; cf. Lemma 2.1. We achieve control over the abscissa of convergence by way of expressing the zeta functions of the groups $\mathbf{H}^m(\mathcal{O}_{\mathfrak{p}})$ in terms of Poincaré series akin to (2.1). It may also be expressed in terms of \mathfrak{p} -adic integrals such as (2.3), but with the families of (\mathcal{O}_K) -polynomials $F_j(\mathbf{Y})$ and $G_i(\mathbf{Y})$ replaced by $\pi^{C(F_j, m)} F_j(\mathbf{Y})$ and $\pi^{C(G_i, m)} G_i(\mathbf{Y})$ for suitable $C(F_j, m), C(G_i, m) \in \mathbb{N}_0$. Whilst the explicit formulae for these “deformed” \mathfrak{p} -adic integrals may differ substantially from the generic formulae, their abscissae of convergence may be bounded in terms of the formulae of Denef type describing the generic case.

Let (Y, h) be a principalization for the \mathcal{O}_K -ideal $\mathcal{I} = \prod_{j \in [w]} (F_j(\mathbf{Y})) \prod_{i \in [v]} (G_i(\mathbf{Y}))$. We continue to use the notation developed in this section so far in this specific context. Recall, in particular, from Section 3.1 the choice of triangulation $\{R_i\}_{i \in [w]_0}$ of $\mathbb{R}_{\geq 0}^{t+1}$ consisting of relatively open, pairwise disjoint simple rational polyhedral cones R_i , $i \in [w]$, and the associated notation. The following is a direct consequence of Proposition 3.2.

Proposition 3.4. *Assume that (Y, h) has good reduction modulo \mathfrak{p} and that \mathfrak{p} does not divide $|\Lambda : M|$ or $|\iota(\Lambda') : \Lambda'|$. There exist $A_j, B_j \in \mathbb{Q}$, for $j \in [z]$, such that the Poincaré series (2.1) satisfies*

$$(3.13) \quad \mathcal{P}_{\mathcal{R}, \mathcal{S}, \mathcal{O}_{\mathfrak{p}}}(s) = 1 + \frac{(1 - q^{-1})^d}{q^{\binom{d}{2}} \prod_{i=1}^{d-1} (1 - q^{-i})} \sum_{i \in W'} c_i(\mathfrak{o}/\mathfrak{p})(q-1)^{|U_i|} \prod_{j \in M_i} \frac{q^{-(A_j s + B_j)}}{1 - q^{-(A_j s + B_j)}}.$$

Proof. Applying equations (2.2), (3.12), and (3.9), we obtain

$$\begin{aligned} \mathcal{P}_{\mathcal{R}, \mathcal{S}, \mathcal{O}_{\mathfrak{p}}}(s) &= 1 + (1 - q^{-1})^{-1} \mathcal{Z}_{\mathcal{O}_{\mathfrak{p}}}(-s/2, -1, us + v - d - 1) \\ &= 1 + (1 - q^{-1})^{-1} \frac{1}{\prod_{i=1}^{d-1} (1 - q^{-i})} Z(\mathbf{a}s + \mathbf{b}) \\ &= 1 + \frac{(1 - q^{-1})^d}{q^{\binom{d}{2}} \prod_{i=1}^{d-1} (1 - q^{-i})} \sum_{i \in W'} c_i(\mathfrak{o}/\mathfrak{p})(q-1)^{|U_i|} \prod_{j \in M_i} \frac{q^{-(A_j s + B_j)}}{1 - q^{-(A_j s + B_j)}}. \end{aligned}$$

for suitable rational numbers A_j and B_j . □

Remark 3.5. By way of construction of the relevant integral, the numbers $c_i(\mathfrak{o}/\mathfrak{p})$ are all divisible by $q^{\binom{d-1}{2}} \prod_{i=1}^{d-1} \frac{(1-q^{-i})}{1-q^{-1}} = |\mathrm{GL}_d(\mathbb{F}_q)/B(\mathbb{F}_q)|/|\mathbb{P}^{d-1}(\mathbb{F}_q)|$. This is due to the fact that we are not using the full generality of integrals of the form (3.1), but restrict to integrands which only involve the matrices' first columns.

Remark 3.6. The numbers A_j are positive for all $j \in [z] \cap W'$. By definition, the set W' comprises exactly those i with the property that there exists $j \in M_i$ with $j \in W'$. Geometrically speaking, this conditions means that at least one of the boundary rays of the simple cone R_i does not lie on the boundary component $\mathbb{R}_{\geq 0}^t \times \{0\}$ of the positive orthant. Consequently, $\sum_{j \in M_i} A_j > 0$ for $i \in W'$.

Corollary 3.7. *Assume that (Y, h) has good reduction modulo \mathfrak{p} and that p does not divide $|\Lambda : M|$ or $|\iota(\Lambda') : \Lambda'|$. The real parts of the poles of $\mathcal{P}_{\mathcal{R}, S, \mathcal{O}_{\mathfrak{p}}}(s)$ are elements of the finite set*

$$(3.14) \quad \mathbf{P} = \left\{ \frac{-B_j}{A_j} \mid j \in [z] \cap W', \right\} \subset \mathbb{Q}.$$

Proof. This follows from inspection of (3.13). \square

Combining formulae (2.2) and (3.11) for the specific integral under consideration yields – at least in principal – explicit formulae of Denef type for the Poincaré series also in the case of bad reduction, albeit not as concise as the one in Proposition 3.4. We have, however, some control over their poles.

Proposition 3.8. *For every nonzero prime ideal \mathfrak{p} of \mathcal{O} such that p does not divide $|\Lambda : M|$ or $|\iota(\Lambda') : \Lambda'|$, the real parts of the poles and thus, in particular, the abscissa of convergence of the Poincaré series $\mathcal{P}_{\mathcal{R}, S, \mathcal{O}_{\mathfrak{p}}}(s)$ are elements of \mathbf{P} .*

Proof. By Corollary 3.7 we may concentrate on the case that the principalization (Y, h) does not have good reduction modulo \mathfrak{p} . As in the generic case, the Poincaré series $\mathcal{P}_{\mathcal{R}, S, \mathcal{O}_{\mathfrak{p}}}(s)$ is obtained from the multivariate zeta function $\mathcal{Z}_{\mathfrak{o}}(\rho, \sigma, \tau)$ defined in (2.3) upon an affine linear substitution of variables; cf. (2.2). In Proposition 3.3 we gave a formula of Denef type for such multivariable \mathfrak{p} -adic integrals in terms of the series $\Xi_{U, (d_{\kappa l})}^N(q, \mathbf{s})$; see (3.10). [3, Proposition 4.5] asserts that, for any $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^l$, the real parts of the poles of the univariable functions $\Xi_{U, (d_{\kappa l})}^N(q, \mathbf{a}\mathbf{s} + \mathbf{b})$ are independent of q , N , and $(d_{\kappa l})$. The claim follows. \square

Proposition 3.9. *For every nonzero prime ideal \mathfrak{p} of \mathcal{O} , the abscissa of convergence of $\zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s)$ is an element of \mathbf{P} .*

Proof. We first consider the case that p does not divide $2|\Lambda : M||\iota(\Lambda') : \Lambda'|$. By Proposition 2.2, in this case the zeta function $\zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s)$ is given by the Poincaré series $\mathcal{P}_{\mathcal{R}, S, \mathcal{O}_{\mathfrak{p}}}(s)$. The abscissa of convergence of the latter is an element of \mathbf{P} by Proposition 3.8.

Assume now that p does divide $2|\Lambda : M||\iota(\Lambda') : \Lambda'|$. In this case, $M(\mathcal{O}_{\mathfrak{p}})$ may not even be a Lie lattice, but there exists $C = C(\mathfrak{p}) \in \mathbb{N}_0$ such that, for all $m \geq C$, the

finitely generated nilpotent pro- p group $\mathbf{H}^m(\mathcal{O}_{\mathfrak{p}}) := \exp(\mathfrak{p}^m M(\mathcal{O}_{\mathfrak{p}}))$ is amenable to the Kirillov orbit method. Fix such an m . The zeta function $\zeta_{\mathbf{H}^m(\mathcal{O}_{\mathfrak{p}})}(s)$ is expressible, via a Poincaré series, in terms of a \mathfrak{p} -adic integral of the form (2.3). In fact, the analysis of Section 2.1 is applicable, with an $\mathcal{O}_{\mathfrak{p}}$ -basis for $\mathfrak{p}^m M(\mathcal{O}_{\mathfrak{p}})$ obtained by multiplying $\mathbf{e} \otimes \mathcal{O}_{\mathfrak{p}}$ by π^m . The relevant \mathfrak{p} -adic integral (2.3) differs from the generic one considered in the previous case essentially in that the families of polynomials $F_j(\mathbf{Y})$ and $G_i(\mathbf{Y})$ are replaced by suitable π -power multiples thereof, viz. $\pi^{C(F_j, m)} F_j(\mathbf{Y})$ and $\pi^{C(G_i, m)} G_i(\mathbf{Y})$ for $C(F_j, m), C(G_i, m) \in \mathbb{N}_0$. Arguing as in the proof of Proposition 3.8, one sees that this leaves unaffected the abscissa of convergence of the relevant univariable substitution. It follows that the abscissa of convergence of the relevant Poincaré series, and thus of $\zeta_{\mathbf{H}^m(\mathcal{O}_{\mathfrak{p}})}(s)$ is an element of \mathbf{P} . But $\alpha(\mathbf{H}^m(\mathcal{O}_{\mathfrak{p}})) = \alpha(\mathbf{G}(\mathcal{O}_{\mathfrak{p}}))$ by Lemma 2.1, so the result follows. \square

4. PROOF OF THEOREM A

Let K be a number field with ring of integers \mathcal{O}_K and Λ be a nilpotent \mathcal{O}_K -lattice of nilpotency class c as in Section 2. Let further L be a finite extension of K , with ring of integers $\mathcal{O} = \mathcal{O}_L$. The group $\mathbf{G}(\mathcal{O}) = \mathbf{G}_{\Lambda}(\mathcal{O})$ is a \mathcal{T} -group of nilpotency class c ; cf. [25, Section 2]. Recall the Euler product (1.3). Choose an \mathcal{O}_K -submodule M of Λ of finite index in Λ as in Section 3.3 and let (Y, h) be a principalization for the \mathcal{O}_K -ideal $\mathcal{I} = \prod_{j \in [u]} (F_j(\mathbf{Y})) \prod_{i \in [v]} (G_i(\mathbf{Y}))$, where $F_j(\mathbf{Y})$ and $G_i(\mathbf{Y})$ are the finite sets of polynomials defined in (2.4) and (2.5). (Recall that they are defined over \mathcal{O}_K as the local bases are obtained from an \mathcal{O}_K -basis for M .) Let Q be the finite set of prime ideals \mathfrak{p} of \mathcal{O} satisfying the following:

- the residue characteristic p of \mathfrak{p} divides $|\Lambda : M| |\iota(\Lambda') : \Lambda'|$,
- (Y, h) does not have good reduction modulo \mathfrak{p} , and
- $(p, c) = (2, 3)$.

We fix a finite triangulation $\{R_i\}_{i \in [w]_0}$ of $\mathbb{R}_{\geq 0}^{t+1}$ as in Section 3 and recall the notation introduced there. Given a prime ideal $\mathfrak{p} \notin Q$ and $i \in W'$ we write, in the notation of Proposition 3.4,

$$Z_{i, \mathfrak{p}}(s) = \frac{(1 - q^{-1})^d}{\prod_{i=1}^{d-1} (1 - q^{-i})} c_i(\mathfrak{o}/\mathfrak{p}) q^{-\binom{d}{2}} (q - 1)^{|U_i|} \prod_{j \in M_i} \frac{q^{-(A_j s + B_j)}}{1 - q^{-(A_j s + B_j)}}.$$

Note that the $Z_{i, \mathfrak{p}}(s)$ are ordinary generating functions in q^{-s} with nonnegative coefficients. Propositions 2.2 and 3.4 amount to the statement that

$$(4.1) \quad \prod_{\mathfrak{p} \notin Q} \zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s) = \prod_{\mathfrak{p} \notin Q} \left(1 + \sum_{i \in W'} Z_{i, \mathfrak{p}}(s) \right).$$

4.1. Proof of (1). Recall that, for *every* $\mathfrak{p} \in \text{Spec}(\mathcal{O})$, the abscissa of convergence $\alpha(\mathbf{G}(\mathcal{O}_{\mathfrak{p}}))$ is an element of the finite set \mathbf{P} of rational numbers defined in (3.14); cf. Proposition 3.9. To prove the first part of Theorem A it thus suffices to show that the abscissa of convergence of $\prod_{\mathfrak{p} \notin Q} \zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s)$ is a rational number which is strictly

larger than $\max \mathbf{P}$ and only depends on \mathbf{G} . Equivalently, it suffices to prove that the abscissa of convergence of

$$\sum_{i \in W'} \left(\sum_{\mathfrak{p} \notin Q} Z_{i,\mathfrak{p}}(s) \right)$$

is such a rational number.

Recall that T is the finite set of irreducible components E_u of the pre-image under h of the subvariety of GL_d/B defined by \mathcal{I} . Let $U \subseteq T$ and $\mathfrak{p} \notin Q$. Recall from (3.8) that, given $i \in W'$ with $U = U_i$, if $i \in W'_U$, then $c_i(\mathfrak{o}/\mathfrak{p}) = c_U(\mathfrak{o}/\mathfrak{p})$, the number of $\mathfrak{o}/\mathfrak{p}$ -rational points of $\overline{E_U} \setminus \cup_{V \supsetneq U} \overline{E_V}$. Let d_U be the dimension of $E_U = \cap_{u \in U} E_u$. Note that $d_U = \binom{d}{2} - |U|$ since the E_u , $u \in T$, are hypersurfaces in a $\binom{d}{2}$ -dimensional variety intersecting with normal crossings; cf. [8, Proposition 4.13].

Let $\{F_{U,b}\}_{b \in I_U}$ be the irreducible components over K of E_U of maximal dimension d_U . For $b \in I_U$ let $l_{\mathfrak{p}}(F_{U,b})$ be the number of irreducible components of $\overline{F_{U,b}}$ over $\mathfrak{o}/\mathfrak{p}$ which are absolutely irreducible over an algebraic closure of $\mathfrak{o}/\mathfrak{p}$. We record the following consequence of the Lang-Weil estimate.

Proposition 4.1. [8, Proposition 4.9] *There exists $C \in \mathbb{R}_{>0}$ such that, for all $U \subseteq T$ and $\mathfrak{p} \notin Q$,*

$$|c_U(\mathfrak{o}/\mathfrak{p}) - \sum_{b \in I_U} l_{\mathfrak{p}}(F_{U,b}) q^{d_U}| \leq C \cdot q^{d_U-1/2}.$$

Moreover, for each $b \in I_U$, $l_{\mathfrak{p}}(F_{U,b}) > 0$ for a set of prime ideals of positive density.

Note that the finitely many primes excluded in [8, Proposition 4.9] are those for which $E_I \setminus \bigcup_{J \supsetneq I} E_J$ has bad reduction; cf. the proof of [8, Lemma 4.7]. Such prime ideals have been excluded from our discussion by the definition of Q .

Corollary 4.2. *For any sequence $(r_{\mathfrak{p}})_{\mathfrak{p} \notin Q}$ of rational numbers, $\sum_{\mathfrak{p} \notin Q} c_U(\mathfrak{o}/\mathfrak{p}) q^{-d_U} r_{\mathfrak{p}}$ converges absolutely if and only if $\sum_{\mathfrak{p} \notin Q} \sum_{b \in I_U} l_{\mathfrak{p}}(F_{U,b}) r_{\mathfrak{p}}$ converges absolutely.*

Similar to [8, Lemma 4.11 and Corollary 4.14] one uses Corollary 4.2 to show that, given $i \in W'$, the abscissa of convergence α_i of $\sum_{\mathfrak{p} \notin Q} Z_{i,\mathfrak{p}}(s)$ – or, equivalently, of $\prod_{\mathfrak{p} \notin Q} (1 + Z_{i,\mathfrak{p}}(s))$ – is

$$\alpha_i = \max \left\{ \frac{1 - \sum_{k \in M_i} B_k}{\sum_{k \in M_i} A_k}, -\frac{B_j}{A_j} \mid j \in M_i, A_j \neq 0 \right\} \in \mathbb{Q}.$$

Recall that $\sum_{k \in M_i} A_k \neq 0$ for $i \in W'$; cf. Remark 3.6. If $i \in [z]$, then $M_i = \{i\}$. Thus

$$\alpha_i = \frac{1 - B_i}{A_i} \text{ for } i \in [z] \cap W'.$$

As $\alpha(\mathbf{G}(\mathcal{O}_{\mathfrak{p}})) \in \mathbf{P}$ for all $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O})$, it follows that

$$\alpha(\mathbf{G}(\mathcal{O})) = \max\{\{\alpha_i \mid i \in W'\} \cup \{\alpha(\mathbf{G}(\mathcal{O}_{\mathfrak{p}})) \mid \mathfrak{p} \in Q\}\} = \max\{\alpha_i \mid i \in W'\} \in \mathbb{Q}.$$

Similar to [8, Corollary 4.14, Lemma 4.15] we obtain that

$$\alpha(\mathbf{G}(\mathcal{O})) = \max \left\{ \frac{1 - B_i}{A_i} \mid i \in [z] \cap W' \right\},$$

and $\alpha_i < \alpha(\mathbf{G}(\mathcal{O}))$ if $i > z$. Note that $\alpha(\mathbf{G}(\mathcal{O})) > \max \mathbf{P}$. Evidently, $a(\mathbf{G}) := \alpha(\mathbf{G}(\mathcal{O}))$ is independent of \mathcal{O} . This concludes the proof of (1).

4.2. Proofs of (2) and the last part. In order to prove part (2) of Theorem A, it suffices to prove the analogous statement for the Euler product (4.1) whose abscissa of convergence is $a(\mathbf{G})$, as we have shown in Section 4.1. Indeed, the finitely many remaining Euler factors of $\zeta_{\mathbf{G}(\mathcal{O})}(s)$ all have abscissa of convergence strictly smaller than $a(\mathbf{G})$ and, as nonzero generating functions in q^{-s} with nonnegative coefficients, do not vanish in a neighbourhood of $s = a(\mathbf{G})$. To establish meromorphic continuation of (4.1) to $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > a(\mathbf{G}) - \delta\}$ for some $\delta > 0$, only depending on \mathbf{G} , we proceed as in [8, Theorem 4.16]. For $\mathfrak{p} \notin Q$, set

$$(4.2) \quad \mathcal{R} = \left\{ i \in [z] \cap W' \mid \frac{1 - B_i}{A_i} = a(\mathbf{G}) \right\},$$

$$V_{\mathfrak{p}}(s) = \prod_{i \in \mathcal{R}} (1 - c_i(\mathfrak{o}/\mathfrak{p}) q^{-d_{U_i}} q^{-(A_i s + B_i)}).$$

Informally speaking, the set \mathcal{R} records which of the rays R_i , $i \in [z]$, contribute to the global abscissa of convergence. The inverses of the functions $V_{\mathfrak{p}}$ will serve as “approximations” of the Euler factors $\zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s)$ for $\mathfrak{p} \notin Q$.

The proof of (2) will be accomplished once we produce constants $\delta_1, \delta_2 \in \mathbb{R}_{>0}$, both depending only on \mathbf{G} , such that the following hold:

- (I) The product $\prod_{\mathfrak{p} \notin Q} V_{\mathfrak{p}}(s)$ has abscissa of convergence $a(\mathbf{G})$ and meromorphic continuation to $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > a(\mathbf{G}) - \delta_1\}$.
- (II) The Euler product $\prod_{\mathfrak{p} \notin Q} (1 + \sum_{i \in W'} Z_{i,\mathfrak{p}}(s)) V_{\mathfrak{p}}(s)$ converges on $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > a(\mathbf{G}) - \delta_2\}$.

Given $i \in \mathcal{R}$, set

$$V_i(s) := \prod_{\mathfrak{p} \notin Q} (1 - c_i(\mathfrak{o}/\mathfrak{p}) q^{-d_{U_i}} q^{-(A_i s + B_i)}),$$

so

$$\prod_{\mathfrak{p} \notin Q} V_{\mathfrak{p}}(s) = \prod_{i \in \mathcal{R}} V_i(s).$$

In order to prove (I), it suffices to show that, for each $i \in \mathcal{R}$, the function $V_i(s)$ defines a meromorphic function on $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > a(\mathbf{G}) - \delta^{(i)}\}$ for some $\delta^{(i)} > 0$ depending only on \mathbf{G} . Fix $i \in \mathcal{R}$, set $U = U_i$, and let $\{F_{U,b}\}_{b \in I_U}$ be as in Section 4.1. Set

$$\tilde{V}_i(s) = \prod_{b \in I_U} \prod_{\mathfrak{p} \notin Q} (1 - l_{\mathfrak{p}}(F_{U,b}) q^{-(A_i s + B_i)}).$$

The following result follows from the straightforward generalization of [8, Lemma 4.6] to arbitrary number fields and the fact that the constant δ in this lemma can be chosen to be $\frac{1}{2}$.

Proposition 4.3. *The function $\tilde{V}_i(s)$ converges on $\left\{s \in \mathbb{C} \mid \operatorname{Re}(s) > \frac{1-B_i}{A_i}\right\}$ and can be continued to a meromorphic function on $\left\{s \in \mathbb{C} \mid \operatorname{Re}(s) > \frac{1-2B_i}{2A_i}\right\}$.*

Note that $\frac{1-2B_i}{2A_i} < \frac{1-B_i}{A_i}$ as $A_i > 0$. The following is a variation of notation introduced in the proof of [8, Theorem 4.16].

Definition 4.4. *Let $(F_{\mathfrak{p}}(s))_{\mathfrak{p} \notin Q}$ and $(G_{\mathfrak{p}}(s))_{\mathfrak{p} \notin Q}$ be families of functions in a complex variable s . Given $\Delta \in \mathbb{R}_{>0}$, we say that the respective Euler products are Δ -equivalent, written*

$$\prod_{\mathfrak{p} \notin Q} F_{\mathfrak{p}}(s) \equiv_{\Delta} \prod_{\mathfrak{p} \notin Q} G_{\mathfrak{p}}(s),$$

if $\sum_{\mathfrak{p} \notin Q} (F_{\mathfrak{p}}(s) - G_{\mathfrak{p}}(s))$ converges on $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > \Delta\}$.

The Lang-Weil estimate (Proposition 4.1) yields that there exist uniform (i.e. independent of \mathfrak{p}) constants $\Delta^{(i)} \in \mathbb{Q}_{>0}$ with $\Delta^{(i)} < \alpha_i (= (1 - B_i)/A_i)$ such that $V_i(s) \equiv_{\Delta^{(i)}} \tilde{V}_i(s)$. Set

$$\delta_1 = a(\mathbf{G}) - \max \left\{ \frac{1 - 2B_i}{2A_i}, \Delta^{(i)} \mid i \in \mathcal{R} \right\} > 0.$$

It follows that $\prod_{\mathfrak{p} \notin Q} V_{\mathfrak{p}}(s)$ is a meromorphic function on $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > a(\mathbf{G}) - \delta_1\}$, which proves (I).

Set

$$d_1 = \max\{0, \alpha_i \mid i \in ([z] \cap W') \setminus \mathcal{R}\} < a(\mathbf{G}).$$

Then, by the definition (4.2) of \mathcal{R} ,

$$\prod_{\mathfrak{p} \notin Q} \left(1 + \sum_{i \in W'} Z_{i,\mathfrak{p}}(s) \right) \equiv_{d_1} \prod_{\mathfrak{p} \notin Q} \left(1 + \sum_{i \in \mathcal{R}} Z_{i,\mathfrak{p}}(s) \right).$$

Recall that if $i \in \mathcal{R}$, then $M_i = \{i\}$ and thus

$$Z_{i,\mathfrak{p}}(s) = \frac{(1 - q^{-1})^d}{\prod_{i=1}^{d-1} (1 - q^{-i})} c_i(\mathfrak{o}/\mathfrak{p}) q^{-\binom{d}{2}} (q - 1)^{|U_i|} \frac{q^{-(A_i s + B_i)}}{1 - q^{-(A_i s + B_i)}}.$$

For $\mathfrak{p} \notin Q$, and $i \in \mathcal{R}$, set

$$\overline{Z_{i,\mathfrak{p}}(s)} = c_i(\mathfrak{o}/\mathfrak{p}) q^{-d v_i} q^{-(A_i s + B_i)}.$$

There exists a uniform constant $d_2 \in \mathbb{Q}_{>0}$ with $d_2 < a(\mathbf{G})$ such that

$$\prod_{\mathfrak{p} \notin Q} \left(1 + \sum_{i \in \mathcal{R}} Z_{i,\mathfrak{p}}(s) \right) \equiv_{d_2} \prod_{\mathfrak{p} \notin Q} \left(1 + \sum_{i \in \mathcal{R}} \overline{Z_{i,\mathfrak{p}}(s)} \right).$$

Set

$$d_3 = \max \left\{ d_1, d_2, \frac{1 - \sum_{k \in J} B_k}{\sum_{k \in J} A_k}, \frac{1 - 2B_j}{2A_j} \mid j \in \mathcal{R}, J \subseteq \mathcal{R}, |J| \geq 2 \right\} < a(\mathbf{G}).$$

Then

$$\prod_{\mathfrak{p} \notin Q} V_{\mathfrak{p}}(s) \equiv_{d_3} \prod_{\mathfrak{p} \notin Q} \left(1 - \sum_{i \in \mathcal{R}} \overline{Z_{i,\mathfrak{p}}(s)} \right),$$

and consequently

$$\begin{aligned} \prod_{\mathfrak{p} \notin Q} \left(1 + \sum_{i \in W'} Z_{i,\mathfrak{p}}(s) \right) V_{\mathfrak{p}}(s) &\equiv_{d_3} \prod_{\mathfrak{p} \notin Q} \left(1 + \sum_{i \in \mathcal{R}} Z_{i,\mathfrak{p}}(s) \right) V_{\mathfrak{p}}(s) \\ &\equiv_{d_3} \prod_{\mathfrak{p} \notin Q} \left(1 + \sum_{i \in \mathcal{R}} \overline{Z_{i,\mathfrak{p}}(s)} \right) \left(1 - \sum_{i \in \mathcal{R}} \overline{Z_{i,\mathfrak{p}}(s)} \right). \end{aligned}$$

The product on the right hand side converges on $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > d_3\}$. Hence (II) follows.

It is clear that $\delta(\mathbf{G}) := \max\{a(\mathbf{G}) - \delta_1, d_3\}$ is independent of the number field K and the zeta function $\zeta_{\mathbf{G}(\mathcal{O})}(s)$ may be continued to $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > a(\mathbf{G}) - \delta(\mathbf{G})\}$. The order $\beta(\mathbf{G})$ of the pole of $\prod_{\mathfrak{p} \notin Q} V_{\mathfrak{p}}^{-1}(s)$ at $s = a(\mathbf{G})$ is clearly an invariant of \mathbf{G} , which proves (2). The statement about the holomorphy of the continued function follows from the fact that the meromorphic continuation was achieved by writing $\zeta_{\mathbf{G}(\mathcal{O})}(s)$ as a product of translates of Artin L -functions (implicit in the proof of Proposition 4.3) and a Dirichlet series convergent on $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > a(\mathbf{G}) - \delta(\mathbf{G})\}$; cf. proof of [8, Corollary 4.22]. The last part of Theorem A follows from the Tauberian theorem [8, Theorem 4.20]. This completes the proof of Theorem A.

5. PROOF OF COROLLARY B

Let G be a finitely generated nilpotent group of nilpotency class c and Hirsch length h , say, and $G_0 \leq G$ a torsion-free subgroup of finite index in G . By the general theory of the Mal'cev correspondence between \mathcal{T} -groups and torsion-free nilpotent Lie rings, there exists an h -dimensional \mathbb{Q} -Lie algebra $L_{G_0}(\mathbb{Q})$ associated to G_0 , together with an injective map $\log : G_0 \rightarrow L_{G_0}(\mathbb{Q})$ such that $\log(G_0)$ rationally spans $L_{G_0}(\mathbb{Q})$; cf. [23, Chapter 6]. If H is a subgroup of finite index in G_0 , then $L_H(\mathbb{Q}) = L_{G_0}(\mathbb{Q})$. In general, $\log(G_0)$ is not even an additive subgroup of $L_{G_0}(\mathbb{Q})$. There exists, however, a subgroup H of finite index in G_0 , and hence in G , such that $\log(H)$ is a \mathbb{Z} -Lie lattice inside $L_{G_0}(\mathbb{Q})$ satisfying $\log(H)' \subseteq c! \log(H)$; cf. [11, Theorem 4.1]. Hence H may be viewed as the group $\mathbf{G}(\mathbb{Z})$ of \mathbb{Z} -points of the group scheme \mathbf{G} associated to the \mathbb{Z} -Lie lattice $\log(H)$. Since H is of finite index in G , it follows from [24, Corollary 4.14] that $\alpha(G) = \alpha(H)$. Moreover, $\zeta_{G,p}(s) = \zeta_{H,p}(s)$ for all primes p which do not divide the index $|G : H|$. The Euler products $\zeta_G(s) = \prod_{p \text{ prime}} \zeta_{G,p}(s)$ and $\zeta_H(s) = \prod_{p \text{ prime}} \zeta_{H,p}(s)$ therefore coincide apart from finitely many Euler factors. Notice that, for every prime p , the abscissae of convergence of $\zeta_{G,p}(s)$ and $\zeta_{H,p}(s)$ coincide (cf. Lemma 2.1)

and are strictly smaller than the abscissa of convergence $\alpha(H)$. All Euler factors are nonzero rational generating functions in p^{-s} (cf. [12, Theorem 1.5]) with nonnegative coefficients. It follows that they do not vanish in a neighbourhood of $s = \alpha(G)$. Corollary B follows now by applying Theorem A to the \mathbb{Z} -Lie lattice $\log(H)$.

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